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CORRELATION FUNCTION ESTIMATOR PERFORMANCE IN NON-GAUSSIAN SPHERICALLY INVARIANT RANDOM PROCESSES

James H. Michels

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In this report, analytic expressions are developed for the variance, error variance and bias of the time-averaged correlation function estimator for stationary, discrete, non-Gaussian complex processes. The expressions derived here pertain to the general class of non-Gaussian processes known as Spherically Invariant Random Processes (SIRP's). Specific results are shown for K-distributed processes which form a special case of the SIRP's. Furthermore, these equations are derived for the general case of processes with unconstrained quadrature components; i.e., for processes exhibiting ellipitical symmetry. For the special case of complex processes with constrained correlation between the quadrature components (i.e., circular symmetry), the resulting analytic expressions attain a simplified form. Validity of the analytic expressions is presented using Monte-Carlo simulations.

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1.0 INTRODUCTION

Ergodicity is the condition which enables time-averaged statistics of random processes to approximate those obtained by ensemble averages. This condition is often assumed in estimation and other signal processing problems since a single realization consisting of a window of time samples is most often available in practice. However, the dependence of the ergodic behavior of random processes upon fundamental process characteristics such as its correlation and statistics as well as the resulting implications on estimation performance evaluation is not often considered. For example, the covariance estimate performed over multiple range cells in an airborne radar application in which the ground clutter is non-homogeneous will exhibit non-ergodic behavior. In this case, estimates obtained by averaging time samples of the data within a given range cell will not be equivalent to those obtained by averaging over range cells.

For the class of non-Gaussian processes known as spherically invariant random processes (SIRP's), the ergodic property does not hold; i.e., the estimates obtained from time-averages are not equivalent to those obtained from ensemble averages. SIRP's are generalizations of the Gaussian random process in that the PDF of every random vector obtained by sampling a SIRP is uniquely determined by the specification of a mean vector, a covariance matrix and a characteristic first order PDF. In addition, the PDF of a random vector obtained by sampling an SIRP is a monotonically decreasing function of a non-negative quadratic form. However, the PDF does not necessarily involve an exponential dependence on the quadratic form, as in the Gaussian case. We also note that many of the attractive properties of the Gaussian random process also apply to the SIRP's. Finally, in the above example of the airborne radar, the SIRP's have been noted to model the statistics of the gound clutter interference [5].

The ergodicity condition for the biased, time-averaged correlation function estimator expressed in terms of the process correlation and statistics (non-Gaussian as well as Gaussian) is derived in this paper. Specifically, analytic expressions are developed for the variance and bias of a time-averaged correlation function estimator for stationary discrete complex spherically invariant random processes (SIRP's) [1]. If the variance of the estimator approaches zero in the limit of infinitely large sample sizes, the ergodic condition holds. However, for SIRP's, it is shown that this condition does not hold. The analytic expressions derived here provide insight regarding both the correlation and statistical parameters which

control the ergodic dependence. Furthermore, they are shown to reduce to the expressions applicable to Gaussian processes as a special case. The development also pertains to the general class of SIRP processes with unconstrained quadrature components (i.e., for processes with elliptical symmetry) where the bandpass processes are, in general, non-stationary [2,3]. For the special case of complex processes with constrained correlation between the quadrature components (i.e., circular symmetry), the resulting analytic expressions attain a simplified form. Validity of the analytic expressions is presented using Monte-Carlo simulations. The affect of the estimator performance when used in a detection scheme designed for SIRP's is reported in [4].

2.0 NON-GAUSSIAN PROCESS DEFINITION

In this section, we discuss the class of non-Gaussian random processes known as Spherically Invariant Random Processes (SIRP's) [1,5]. Following Rangaswamy [5], we first define a spherically invariant random vector (SIRV) as a random vector (real or complex) whose PDF is uniquely determined by the specification of a mean vector, a covariance matrix and a characteristic first order PDF. A spherically invariant random process (SIRP) is a random process (real or complex) such that every random vector obtained by sampling this process is an SIRV. An important theorem in the theory of such processes is the representation theorem [1] stated as follows.

Theorem 1. If a random vector is an SIRV, then there exists a non-negative random variable S such that the PDF of the random vector conditioned on S is a multivariate Gaussian PDF.

For an SIRV, we consider the product

$$\underline{y}_{1,N} = s\underline{x}_{1,N} \tag{2.1}$$

where $\underline{y}_{1,N} = [y_1\ y_2\ \ldots\ y_N]^T$ denotes the SIRV, $\underline{x}_{1,N} = [x_1\ x_2\ \ldots\ x_N]^T$ is a Gaussian random vector with zero mean and covariance matrix Σ and s is a real, non-negative random variable with characteristic PDF $f_S(s)$. Statistical independence between $\underline{x}_{1,N}$ and s is assumed for convenience. In [5], several characterisic PDF's for $f_S(s)$ are considered which provide various PDF's for $f_y(y)$. Among others, they include the Chi, Weibull, Generalized Rayleigh, Rician, the K-distribution, Laplace, Cauchy, Student-t and, as a special case, the Gaussian.

In section 3, we consider K-distributed processes using a form of the Gamma distribution for $f_S(s)$. For processes consisting of in-phase (real) and quadrature (imaginary) components, the K-distributed envelope PDF is expressed as

$$f_R(r) = \frac{2\sqrt{\alpha}}{\Gamma(\alpha)}\left[\frac{\sqrt{\alpha}r}{2}\right]^{\alpha} K_{\alpha-1}(\sqrt{\alpha}r) \qquad (0 \leq r \leq \infty) \tag{2.2}$$

where $\Gamma(\alpha)$ is the Eulerian Gamma function, $K_\alpha(\bullet)$ is the modified Bessel function of the second kind with order α. Here, α is refered to as the shape parameter.

3.0 VARIANCE OF THE TIME-AVERAGED AUTOCORRELATION FUNCTION

The ensemble autocorrelation function is defined as the expectation of lagged products of a given stationary process when averaged over an ensemble of realizations. If this function is equal to the time-averaged autocorrelation function obtained from a single realization, the process is called autocorrelation ergodic. Consider the time-averaged estimate of the autocorrelation function using N_T temporal observation samplesof the i^{th} channel process $y_i(n)$ [6]

$$\hat{R}_y(l,N_T) = \begin{cases} \frac{1}{M} \sum_{n=0}^{N_T-l-1} y_i(n)y_i^*(n-l) & 0 \leq l \leq N_T-1 \\ \frac{1}{M} \sum_{n=0}^{N_T-|l|-1} y_i^*(n)y_i(n-|l|) & -(N_T-1) \leq l \leq 0. \end{cases} \tag{3.1}$$

For $M = N_T$, we obtain the biased estimator $\hat{R}_y(l,N_T)$ while for $M=N_T-l$, we have the unbiased estimator. In this paper, we consider the biased correlation function estimator with limited data since it ensures positive semi-definiteness. The derivation for the unbiased estimator follows directly. A similar derivation also holds for the cross-correlation function using relations developed in [3]. We now define

$$\phi(n,l) = y_i(n)y_i^*(n - l) \tag{3.2a}$$

and

$$R_{\phi\phi}(k,l) = E[\phi(n,l)\phi^*(n - k,l)] \tag{3.2b}$$

so that

$$C_{\phi\phi}(k,l) = E\left[\{\phi(n,l) - E[\phi(n,l)]\}\{\phi^*(n-k,l) - E[\phi^*(n-k,l)]\}\right] \tag{3.2c}$$

$$= R_{\phi\phi}(k,l) - E[\phi(n,l)]E[\phi^*(n-k,l)]. \tag{3.2d}$$

Using eq(3.1), the mean $\bar{R}_y(l,N_T)$ and variance $V_y(l,N_T)$ of $\hat{R}_y(l,N_T)$ can now be expressed as

$$\bar{R}_y(l,N_T) = E[\hat{R}_y(l,N_T)] = \begin{cases} \dfrac{1}{N_T} \displaystyle\sum_{n=0}^{N_T-l-1} R_y(l) & 0 \le l \le N_T - 1 \\ \\ \dfrac{1}{N_T} \displaystyle\sum_{n=0}^{N_T-|l|-1} R_y(l) & -(N_T-1) \le l \le 0. \end{cases} \tag{3.3}$$

where $R_y(l) = E[y_i(n)y_i^*(n-l)]$ and

$$V_y(l,N_T) =$$

$$= E\left\{ [\hat{R}_y(l,N_T) - E[\hat{R}_y(l,N_T)]][\hat{R}_y^*(l,N_T) - E[\hat{R}_y^*(l,N_T)\right\} \tag{3.4a}$$

$$= E[\hat{R}_y(l,N_T)\hat{R}_y^*(l,N_T)] - E[\hat{R}_y(l,N_T)]E[\hat{R}_y^*(l,N_T)] \tag{3.4b}$$

$$= \frac{1}{N_T} \sum_{k=-(N_T-|l|-1)}^{N_T-|l|-1} \left[1 - \frac{|l|+|k|}{N_T}\right] C_{\phi\phi}(k,l). \tag{3.4c}$$

Expanding eq(3.3), we have

$$E[\hat{R}_{yT_b}(l,N_T)] = \begin{cases} R_y(l) - \dfrac{l}{N_T} R_y(l) & 0 \le l \le N_T - 1 \\ \\ R_y(l) - \dfrac{|l|}{N_T} R_y(l) & -(N_T-1) \le l \le 0. \end{cases} \tag{3.5}$$

We note that the second term on the right hand side of eq(3.5) denotes the bias of the estimate. Thus, the magnitude of the bias $|B|$ is expressed as

$$|B| = |l|\,|R_y(l)|/N_T. \tag{3.6}$$

From eq(3.2a)

$$E[\phi(n,l)] = R_y(l) \tag{3.7a}$$

and

$$E[\phi^*(n-k,l)] = R_y^*(l) \tag{3.7b}$$

so that from eq(3.2d)

$$C_{\phi\phi}(k,l) = R_{\phi\phi}(k,l) - |R_y(l)|^2. \tag{3.8}$$

And so, eq(3.4c) becomes

$$V_y(l,N_T) = \frac{1}{N_T} \sum_{k=-(N_T-|l|-1)}^{N_T-|l|-1} \left[1 - \frac{|l|+|k|}{N_T}\right][R_{\phi\phi}(k,l) - |R_y(l)|^2]. \tag{3.9}$$

We first consider $R_{\phi\phi}(k,l)$ in eq(3.9). Using eq(3.2a), we have

$$R_{\phi\phi}(k,l) = E[\phi(n,l)\phi^*(n-k,l)] \tag{3.10a}$$

$$= E[y_i(n)y_i^*(n-l)y_i^*(n-k)y_i(n-l-k)]. \tag{3.10b}$$

From the representation theorem described in section 2, $y_i(n) = sx_i(n)$ where s is a real, non-negative random variable. Since s is statistically independent of $x_i(n)$, eq(3.10b) can be expressed as

$$R_{\phi\phi}(k,l) = E[s^4]E[x_i(n)x_i^*(n-l)x_i^*(n-k)x_i(n-l-k)]. \tag{3.11}$$

The second expectation contains zero-mean, jointly stationary Gaussian quadrature components $x_{iI}(n)$ and $x_{iQ}(n)$. It can therefore be expressed as [see Appendix B]

$$E[x_i(n)x_i^*(n-l)x_i^*(n-k)x_i(n-l-k)] = |R_x(l)|^2 + |R_x(k)|^2 + F_{x_{ii}}(l,k) \tag{3.12}$$

where

$$R_x(l) = E[x_i(n)x_i^*(n-l)] \tag{3.13a}$$

and

$$F_{x_{ii}}(l,k) = \{R_x^{II}(l+k) - R_x^{QQ}(l+k)\}\{R_x^{II}(l-k) - R_x^{QQ}(l-k)\}$$
$$+ \{R_x^{QI}(l+k) + R_x^{IQ}(l+k)\}\{R_x^{QI}(l-k) + R_x^{IQ}(l-k)\}$$
$$- j\{R_x^{II}(l+k) - R_x^{QQ}(l+k)\}\{R_x^{QI}(l-k) + R_x^{IQ}(l-k)\}$$
$$+ j\{R_x^{II}(l-k) - R_x^{QQ}(l-k)\}\{R_x^{QI}(l+k) + R_x^{IQ}(l+k)\}. \quad (3.13b)$$

And so, eq(3.11) becomes

$$R_{\phi\phi}(k,l) = E[s^4]\left\{ |R_x(l)|^2 + |R_x(k)|^2 + F_{x_{ii}}(l,k) \right\}. \quad (3.14)$$

Next, we consider the term $|R_y(l)|^2$ in eq(3.6). Since

$$R_y(l) = E[y_i(n)y_i^*(n - l)] = E[s^2]R_x(l), \quad (3.15)$$

then

$$|R_y(l)|^2 = E^2[s^2]|R_x(l)|^2. \quad (3.16)$$

Using eqs(3.14) and (3.16) in (3.9), we have

$$V_y(l,N_T) = \frac{1}{N_T} \sum_{k=-(N_T-|l|-1)}^{N_T-|l|-1} \left[1 - \frac{|l|+|k|}{N_T}\right] \left\{ E[s^4]\{|R_x(l)|^2 + |R_x(k)|^2 + \mathrm{Re}F_{x_{ii}}(l,k)\} - E^2[s^2]|R_x(l)|^2 \right\} \quad (3.17)$$

where the real part of $F_{x_{ii}}(l,k)$ results due to the cancellation of the imaginary terms in eq(3.13b) when the summation over positive and negative values of k is taken. For circular Gaussian processes, $R_x^{II}(l) = R_x^{QQ}(l)$ and $R_x^{IQ}(l) = - R_x^{QI}(l)$, so that from eq(3.13b) $F_{x_{ii}}(l,k) = 0$ for all l,k. In this case, eq(3.17) can be written as

$$V_y(l,N_T) =$$
$$= \frac{1}{N_T} \sum_{k=-(N_T-|l|-1)}^{N_T-|l|-1} \left[1 - \frac{|l|+|k|}{N_T}\right] \left\{ E[s^4]|R_x(k)|^2 + [E[s^4]-E^2[s^2]]|R_x(l)|^2 \right\}. \quad (3.18)$$

Since $R_x(l)$ does not depend on k, the summation over the second term in the { } bracket results in

$$\frac{1}{N_T}\sum_{k=-(N_T-|l|-1)}^{N_T-|l|-1}\left[1-\frac{|l|+|k|}{N_T}\right]\left\{\{E[s^4]-E^2[s^2]\}|R_x(l)|^2\right\} = \{E[s^4]-E^2[s^2]\}|R_x(l)|^2. \tag{3.19}$$

Thus, eq(3.18) becomes

$$V_y(l,N_T) =$$
$$= \frac{1}{N_T}\sum_{k=-(N_T-|l|-1)}^{N_T-|l|-1}\left[1-\frac{|l|+|k|}{N_T}\right]\left\{E[s^4]|R_x(k)|^2\right\} + \{E[s^4]-E^2[s^2]\}|R_x(l)|^2 \tag{3.20}$$

Eq(3.20) is the significant contribution of this paper. It describes the variance of the time-averaged correlation function estimator for SIRP's. What is most significant is the fact that the second term on the RHS of eq(3.20) is not dependent upon N_T; i.e., as the time window sample size N_T increases, this term does not provide a decrease in $V_y(l,N_T)$.

We now introduce the error variance $EV_y(l,N_T)$ defined as

$$EV_y(l,N_T) =$$
$$= E\left\{[\hat{R}_y(l,N_T) - R_y(l,N_T)]][\hat{R}_y^*(l,N_T) - R_y^*(l,N_T)\right\} \tag{3.21}$$

From eq(3.6), the magnitude of the bias B can be expressed as

$$|B| = \frac{|l|}{N_T}|R_y(l)| = \frac{|l|}{N_T}E[s^2]|R_x(l)| \tag{3.22}$$

and recognizing that the error variance $EV_y(l,N_T)$ can be written as

$$EV_y(l,N_T) = V_y(l,N_T) + |B|^2 \tag{3.23}$$

we have

$$EV_y(l,N_T) =$$

$$= \frac{1}{N_T} \sum_{k=-(N_T-|l|-1)}^{N_T-|l|-1} \left[1 - \frac{|l|+|k|}{N_T}\right] \left\{ E[s^4]|R_x(k)|^2 \right\} + \{E[s^4]-E^2[s^2]\}|R_x(l)|^2$$

$$+ \frac{|l|^2}{N_T^2} E^2[s^2]|R_x(l)|^2 \tag{3.24}$$

For the special case of K-distributed processes with shape parameter α and scale parameter b, it can be shown [see Appendix C] that $E[s^4] = (\alpha)(\alpha+1)/b^4$ and $E[s^2] = \alpha/b^2$. We can let $E[s^2] = 1$ without loss of generality. And so, for $b=\sqrt{\alpha}$, eq(3.20) becomes

$$V_y(l,N_T) = \frac{1}{N_T} \sum_{k=-(N_T-|l|-1)}^{N_T-|l|-1} \left[1 - \frac{|l|+|k|}{N_T}\right] \left\{ \left[1+\frac{1}{\alpha}\right]|R_x(k)|^2 \right\} + \frac{1}{\alpha}|R_x(l)|^2 \tag{3.25}$$

For Gaussian processes, $\alpha \rightarrow \infty$ and eq(3.25) reduces to eq(5.2.8) of [3]; i.e.,

$$V_y(l,N_T) = \frac{1}{N_T} \sum_{k=-(N_T-|l|-1)}^{N_T-|l|-1} |R_x(k)|^2. \tag{3.26}$$

We note that the term $(1/\alpha)|R_x(l)|^2$ in eq(3.25) becomes the dominant term as α is reduced; i.e., as the processes tend toward non-Gaussian with high tails. This term also introduces a stronger dependence of the variance upon the lag value than the corresponding Gaussian case expressed in eq(3.26). This result is also true for the error variance expressed in eq(3.24). What is most significant, however, is the fact that this latter term is not dependent upon N_T; i.e., as the time window sample size N_T increases, this term does not provide a decrease in $V_y(l,N_T)$. This is an indication of the fact that the SIRP processes are non-ergodic. In this case, $V_y(l,N_T)$ does not tend toward zero as N_T approaches infinity. In section 4, we present results which illustrate the significance of this result.

4.0 RESULTS

In this chapter, we validate the analytic expressions developed in section 3 using Monte-Carlo simulation. K-distributed processes with an exponentially shaped correlation function are generated via a first order autoregressive AR(1) process. In this case the white noise driving term is K-distributed. In each case, N_R realizations of the random process $y_i(n)$ are generated with N_T time samples per realization. The time-average estimator is used on each m^{th} realization to compute $R_y(l,N_T|m)$. The sample variance of these estimates are then computed over the N_R realizations using the expressions derived in the next subsection. For N_R large, these values are compared to those obtained via the ensemble averaged expressions in the previous chapter.

4.1 Computed Variance of the Time-Averaged Correlation Functions

Consider N_R realizations of the random process $y_i(n)$. Let each realization be indexed by the integer m; m=1,2,...,N_R. Corresponding to the realization with index m, let $\hat{R}_y(l,N_T|m)$ be the biased, time-averaged cross-correlation function estimate using N_T observation samples. The sample variance of the time-averaged cross-correlation function estimate is computed from N_R realizations using the expression

$$\mathrm{Var}[\hat{R}_y(l,N_T):N_R] = \frac{1}{N_R-1} \sum_{m=1}^{N_R} \left| \hat{R}_y(l,N_T|m) - \bar{\hat{R}}_y(l,N_R|m) \right|^2 \tag{4.1}$$

where

$$\bar{\hat{R}}_y(l,N_R|m) = \frac{1}{N_R} \sum_{m=1}^{N_R} \hat{R}_y(l,N_T|m). \tag{4.2}$$

4.2 Simulation Results

In this section, we validate eq(3.21) using computer generated random processes. Specifically, a scaler autoregressive process of order one AR(1) with a K-distributed white noise driving term was used to synthesize the data time series. The correlation function for this process is exponential and can be expressed in terms of the one-lag temporal correlation parameter λ such that [2]

$$R_y(k) = \sigma_y^2(\lambda)^{|k|}\exp[j\theta(k)] \tag{4.3}$$

where σ_y^2 is the variance of the observation data process y(n) and λ is the one-lag temporal correlation parameter such that $0 \le \lambda \le 1$. In the special case of the AR(1) process used here, $\sigma_y^2 = 4$, $\theta(k)=0$ and $\lambda = -a(1)$ where a(1) is the AR(1) coefficient.

Computed results were obtained using Monte-Carlo simulation with 10,000 realizations. A plot of $V_y(l,N_T)$ versus the shape parameter α is shown in Figure1 for lag values $l=0,1$ and $N_T=100$. These curves reveal that for $\alpha \le 5$ (i.e., for non-Gaussian SIRP's with high tails), $V_y(l,N_T)$ increases dramatically as α decreases.

In Figure 2, analytic values of $V_y(l,N_T)$ are plotted versus the time sample window size, N_T, with α as a parameter. These curves show the effect of the second term on the right hand side of eq(3.21) which is independent of N_T. These results show that the estimator lacks consistency; i.e., $V_y(l,N_T)$ does not decrease with increasing N_T. This results from the non-ergodic property of the SIRP's.

Finally, in Figure 3, analytic and computed values of $V_y(l,N_T)$ versus the one-lag temporal correlation parameter λ are shown for lag values $l=0,1$. The AR(1) process had a shape parameter of $\alpha=0.3$ in this case. We observe that $V_y(l,N_T)$ for $l=0$ is relatively insensitive to changes in the temporal correlation of the processes. Again the second term on the RHS of eq(3.21) accounts for this effect. We also point out that at lag $l=0$, the correlation function estimator is estimating the variance of the process. Thus, the high tails for this process affect the estimate of the variance. For lag values other than zero, however, $V_y(l,N_T)$ is highly dependent on the temporal correlation. This result can also be explained by the second term in eq(3.21) which decreases with decreasing temporal correlation for lag values other than zero.

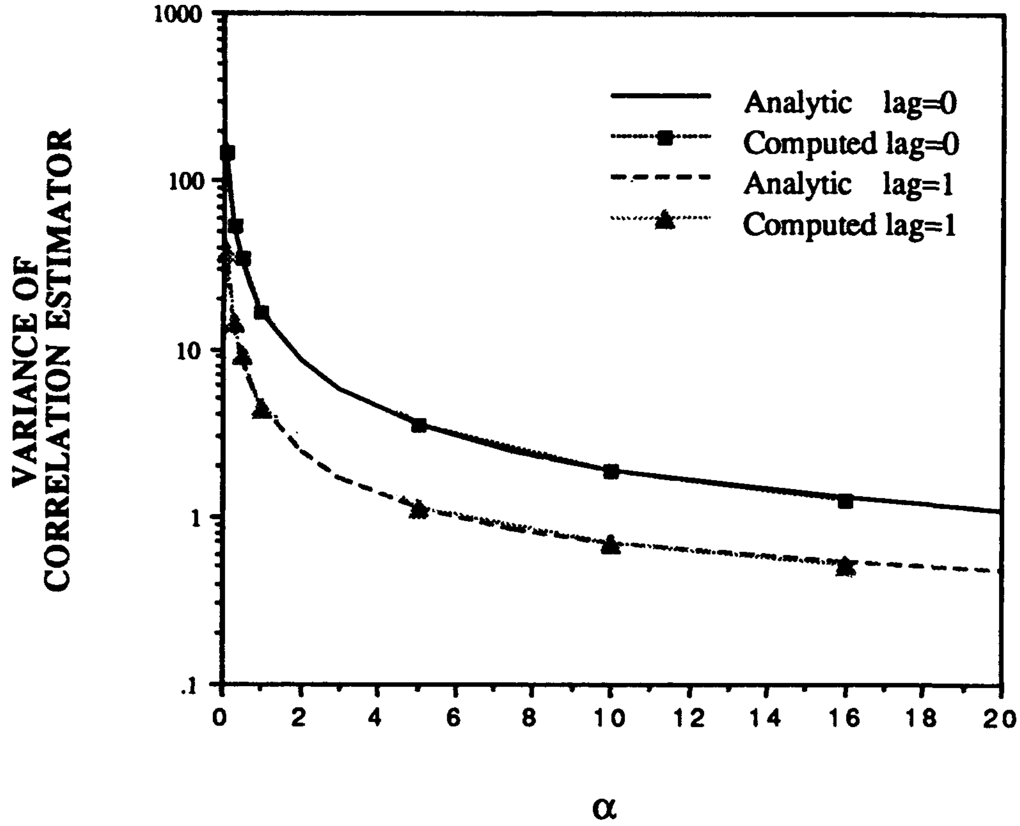

Figure 1 Analytic and computed $V_y(l,N_T)$ versus the shape parameter α for lag values $l = 0, 1$ for an AR(1) K-distributed process with $\sigma_y^2 = 4$, N_T=100.

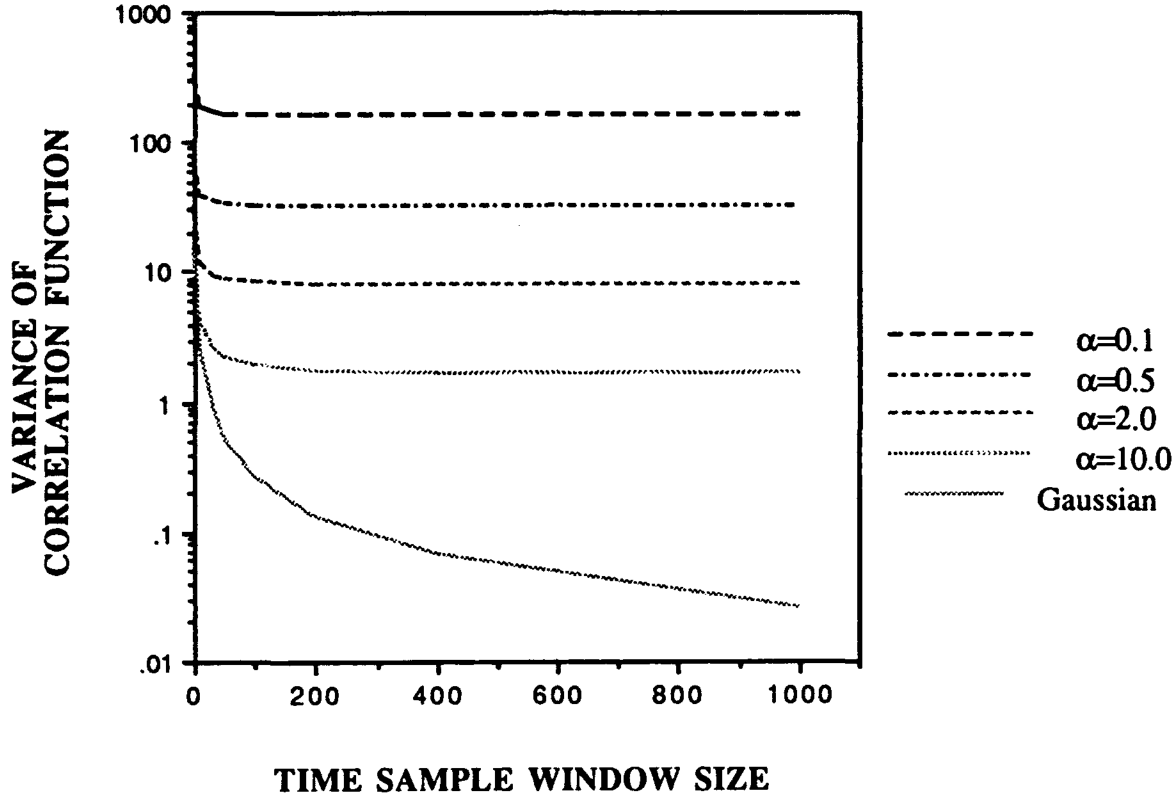

Figure 2 Analytic values of $V_y(l,N_T)$ versus the time sample window size N_T for an AR(1) K-distributed process with $\sigma_y^2 = 4$ and the shape parameter α as a parameter.

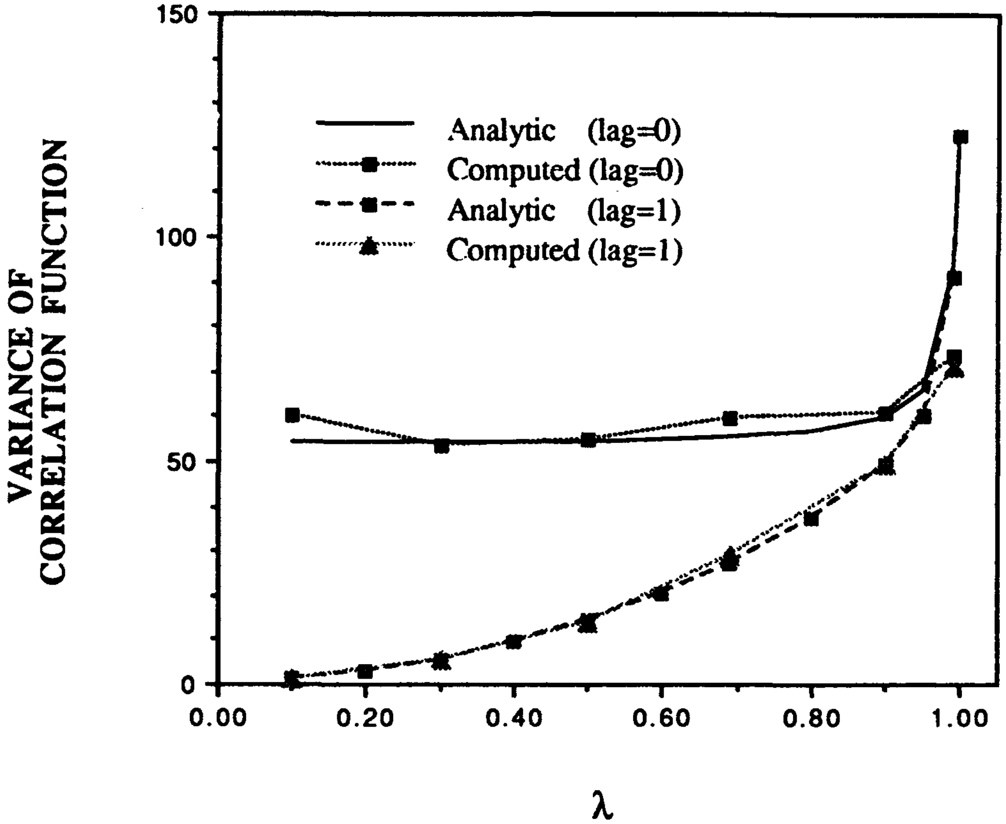

Figure 3 Analytic and computed $V_y(l,N_T)$ versus the one-lag temporal correlation parameter λ for lag values l=0 and 1 using an AR(1) K-distributed process with shape parameter $\alpha = 0.3$, $\sigma_y^2 = 4$, N_T=100.

5.0 CONCLUSIONS

In this paper, general analytic expressions have been derived for the variance, the error variance and the bias of the time-averaged correlation function estimator when observing non-Gaussian spherically invariant random processes (SIRP's). These expressions were validated for the special case of a K-distibuted AR(1) process via a Monte-Carlo computer simulation. Furthermore, these analytic functions were expressed in terms of the process variance, the temporal correlation, the data window sizes used to average the estimates, and the shaping parameter of the K-distibuted process. Finally, they were shown to reduce to the expressions for Gaussian processes as a special case.

REFERENCES

[1] Yao,K., "A repre :ntation theorem and its applications to spherically invariant random processes," IEEE Transactions on Information Theory, vol. IT-19, pp. 600-608, 1973.

[2] Michels, J. H., "Synthesis of Multichannel Autoregressive Random Processes and Ergodicity Considerations", RADC-TR-90-211, July, 1990.

[3] Michels, J. H., Multichannel Detection Using the Discrete-Time Model-Based Innovations Approach, PhD dissertation, Syracuse University, Syracuse, N.Y., May 1991.

[4] Rangaswamy, M., Weiner, D., Michels, J.H., "Multichannel Detection for Correlated Non-Gaussian Random Processes Based on Innovations", 1993 SPIE International Symposium on Optical Engineering and Photonics, April 12-16, 1993, Orlando, Fl.

[5] Rangaswamy, M., 'Spherically Invariant Random Processes for Radar Clutter Modeling, Simulation and Distributed Identification', PhD dissertation, Syracuse University, 1992.

[6] Marple, S.L., Digital spectral Analysis with Applications, Prentice Hall, NJ, 1987.

[7] Rangaswamy, M., Weiner, D., Michels, J.H., "Innovations Based Detection Algorithm for Correlated Non-Gaussian Random Processes," accepted for presentation at the Sixth SSAP Workshop on Statistical Signal and Array Processing, Victoria, British Columbia, Canada, 7-9 Oct., 1992.

[8] Papoulis, A., Signal Analysis, McGraw-Hill Book Co., 1977.

[9] Jenkins, G., Watts, D., Spectral Analysis and its applications, Holden Day,1968.

[10] Jacovitti, G., Cusani, R., "Performance of Normalized Correlation Estimators for Complex Processes", IEEE Transactions on Signal Processing, vol. 40, no. 1, January 1992.

[11] Jacovitti, G., Neri, A., "Special Techniques for the estimation of the ACF of Spherically Invariant Random Processes", Proceedings of the 4th Annual ASSP Workshop on Spectrum Estimation and Modeling, Minneapolis, Minnesota, August 3-5, 1988.

APPENDIX A DERIVATION OF EQ(3.3)

In this appendix, we derive the expression for the variance of $\hat{R}_y(l,N)$ expressed by eq(3.3). Consider the biased, time-averaged estimate of the autocorrelation function [6]; ie.,

$$\hat{R}_y(l,N_T) = \begin{cases} \dfrac{1}{N_T} \displaystyle\sum_{n=0}^{N_T-l-1} y_i(n) y_i^*(n-l) & 0 \le l \le N_T-1 \\ \\ \dfrac{1}{N_T} \displaystyle\sum_{n=0}^{N_T-\|l\|-1} y_i^*(n) y_i(n-\|l\|) & -(N_T-1) \le l \le 0. \end{cases} \tag{A.1}$$

where the symbol ^ in this discussion designates the quantity as an estimate. Let

$$\phi(n,l) = y_i(n) y_i^*(n-l). \tag{A.2}$$

Assuming stationarity, the covariance of $\phi(n,l)$ can be expressed as

$$C_{\phi\phi}(k,l) = E[\{\phi(n,l)-E[\phi(n,l)]\}\{\phi^*(n-k,l) - E[\phi^*(n-k,l)]\}] \tag{A.3a}$$

$$\begin{aligned} &= E[\phi(n,l)\phi^*(n-k,l)] - E[\phi(n,l)]E[\phi^*(n-k,l)] \\ &\quad - E[\phi(n,l)]E[\phi^*(n-k,l)] + E[\phi(n,l)]E[\phi^*(n-k,l)] \end{aligned} \tag{A.3b}$$

$$= R_{\phi\phi}(k,l) - E[\phi(n,l)]E[\phi^*(n-k,l)] \tag{A.3c}$$

where

$$R_{\phi\phi}(k,l) = E[\phi(n,l)\phi^*(n-k,l)]. \tag{A.4}$$

Assuming stationarity, we have from eq(A.2),

$$E[\phi(n,l)] = R_y(l) \tag{A.5a}$$

and

$$E[\phi^*(n-k,l)] = R_y^*(l) \tag{A.5b}$$

so that

$$C_{\phi\phi}(k,l) = R_{\phi\phi}(k,l) - |R_y(l)|^2 \tag{A.6a}$$

$$= E[y_i(n)y_i^*(n-l)y_i^*(n-k)y_i(n-l-k)] - |R_y(l)|^2. \tag{A.6b}$$

Now consider the variance of the complex time averaged estimate $\hat{R}_{y_{T_b}}(l,N)$ which can be expressed as

$$V_y(l,N_T) = E\left\{ [\hat{R}_y(l,N_T)-E[\hat{R}_y(l,N_T)]][\hat{R}_y^*(l,N_T)-E[\hat{R}_y^*(l,N_T)]] \right\} \tag{A.7a}$$

$$= E[\hat{R}_y(l,N_T)\hat{R}_y^*(l,N_T)] - E[\hat{R}_y(l,N_T)]E[\hat{R}_y^*(l,N_T)]. \tag{A.7b}$$

Using eq(A.1) for positive and negative l, we have

$$\hat{R}_y(l,N_T)\hat{R}_y^*(l,N_T) =$$

$$= \begin{cases} \dfrac{1}{N_T^2} \displaystyle\sum_{n=0}^{N_T-l-1} \sum_{p=0}^{N_T-l-1} y_i(n)y_i^*(n-l)y_i^*(p)y_i(p-l) & 0 \le l \le N_T - 1 \\ \dfrac{1}{N_T^2} \displaystyle\sum_{n=0}^{N_T-|l|-1} \sum_{p=0}^{N_T-|l|-1} y_i^*(n)y_i(n-|l|)y_i(p)y_i^*(p-|l|) & -(N_T-1) \le l \le 0 \end{cases} \tag{A.8}$$

so that

$$E[\hat{R}_y(l,N_T)\hat{R}_y^*(l,N_T)] =$$

$$= \begin{cases} \dfrac{1}{N_T^2} \displaystyle\sum_{n=0}^{N_T-l-1} \sum_{p=0}^{N_T-l-1} E[y_i(n)y_i^*(n-l)y_i^*(p)y_i(p-l)] & 0 \le l \le N_T - 1 \\ \dfrac{1}{N_T^2} \displaystyle\sum_{n=0}^{N_T-|l|-1} \sum_{p=0}^{N_T-|l|-1} E[y_i^*(n)y_i(n-|l|)y_i(p)y_i^*(p-|l|)] & -(N_T-1) \le l \le 0. \end{cases} \tag{A.9}$$

Also,

$$E[\hat{R}_y(l,N_T)] = \begin{cases} \dfrac{1}{N_T}\displaystyle\sum_{n=0}^{N_T-l-1} R_y(l) & 0 \le l \le N_T - 1 \\ \dfrac{1}{N_T}\displaystyle\sum_{n=0}^{N_T-|l|-1} R_y(l) & -(N_T-1) \le l \le 0. \end{cases} \qquad (A.10a)$$

Expanding eq(A.10a), we have

$$E[\hat{R}_y(l,N_T)] = \begin{cases} R_y(l) - \dfrac{l}{N_T}R_y(l) & 0 \le l \le N_T - 1 \\ R_y(l) - \dfrac{|l|}{N_T}R_y(l) & -(N_T-1) \le l \le 0. \end{cases} \qquad (A.10b)$$

We note that the second term on the right hand side of eq(A.10b) denotes the bias of the estimate. Thus, the magnitude of the bias is expressed as $|B| = |l|\ |R_y(l)|/N_T$.

From eq(A.10a), we have

$$E[\hat{R}_y(l,N_T)]E[\hat{R}_y^*(l,N_T)] =$$

$$= \begin{cases} \dfrac{1}{N_T^2}\displaystyle\sum_{n=0}^{N_T-l-1}\sum_{p=0}^{N_T-l-1} |R_y(l)|^2 & 0 \le l \le N_T \\ \dfrac{1}{N_T^2}\displaystyle\sum_{n=0}^{N_T-|l|-1}\sum_{p=0}^{N_T-|l|-1} |R_y(l)|^2 & -(N_T-1) \le l \le 0. \end{cases} \qquad (A.11)$$

Using eqs(A.9) and (A.11) in eq(A.7b), we obtain

$V_y(l,N_T) =$

$$
= \begin{cases} \dfrac{1}{N_T^2} \displaystyle\sum_{n=0}^{N_T-l-} \sum_{p=0}^{N_T-l-1} \{E[y_i(n)y_i^*(n-l)y_i^*(p)y_i(p-l)] - |R_y(l)|^2\} \\ \qquad \text{for } 0 \le l \le N_T - 1 \\ \dfrac{1}{N_T^2} \displaystyle\sum_{n=0}^{N_T-|l|-} \sum_{p=0}^{|N_T-|l|-1} \{E[y_i^*(n)y_i(n-|l|)y_i(p)y_i^*(p-|l|)] - |R_y(l)|^2\} \\ \qquad \text{for } -(N_T-1) \le l \le 0. \end{cases} \tag{A.12}
$$

Using eq(A.6b) in (A.12)

$$
V_y(l,N_T) = \begin{cases} \dfrac{1}{N_T^2} \displaystyle\sum_{n=0}^{N_T-l-1} \sum_{p=0}^{N_T-l-1} C_{\phi\phi}(n-p,l) & 0 \le l \le N_T - 1 \\ \dfrac{1}{N_T^2} \displaystyle\sum_{n=0}^{N_T-|l|-} \sum_{p=0}^{|N_T-|l|-1} C_{\phi\phi}^*(n-p,|l|) & -(N_T-1) \le l \le 0. \end{cases} \tag{A.13}
$$

We now let $k = n - p$ where

$$-(N_T - l - 1) \le k \le N_T - l - 1 \qquad \text{for} \qquad 0 \le l \le N_T - 1 \tag{A.14a}$$

$$-(N_T - |l| - 1) \le k \le N_T - |l| - 1 \qquad \text{for} \qquad -(N_T-1) \le l \le 0. \tag{A.14b}$$

We also note that eq(A.14b) is equivalent to eq(A.14a) for negative values of l so that

$$V_y(l,N_T) =$$

$$= \begin{cases} \dfrac{1}{N_T^2} \displaystyle\sum_{k=-(N_T-|l|-1)}^{N_T-|l|-1} [N_T - |l| - |k|]\, C_{\phi\phi}(k,l) & 0 \le l \le N_T - 1 \\ \dfrac{1}{N_T^2} \displaystyle\sum_{k=-(N_T-|l|-1)}^{N_T-|l|-1} [N_T - |l| - |k|]\, C^*_{\phi\phi}(k,|l|) & -(N_T-1) \le l \le 0. \end{cases} \tag{A.15}$$

However, for negative lag l, we have

$$C^*_{\phi\phi}(k,|l|) = C_{\phi\phi}(k,l) \tag{A.16}$$

so that after dividing the bracketed factor by one of the N_T terms in the denominator

$$V_y(l,N_T) = \frac{1}{N_T} \sum_{k=-(N_T-|l|-1)}^{N_T-|l|-1} \left[1 - \frac{|l| + |k|}{N_T}\right] C_{\phi\phi}(k,l) \tag{A.17}$$

for both positive and negative values of l. In Appendix G of [3], we show that the imaginary terms in $C_{\phi\phi}(k,l)$ cancel when summed over positive and negative values of k so that

$$V_y(l,N_T) = \frac{1}{N_T} \sum_{k=-(N_T-|l|-1)}^{N_T-|l|-1} \left[1 - \frac{|l|+|k|}{N_T}\right] \mathrm{Re}\{C_{\phi\phi}(k,l)\}. \tag{A.18}$$

APPENDIX B DERIVATION OF EQ(3.9)

In this appendix, we derive eq(3.9). Consider the quantity

$$R_{x_{\phi\phi}}(k,l) = E[x_i(n)x_i^*(n-l)x_i^*(n-k)x_i(n-l-k)]. \tag{B.1}$$

In the special case where the process $x_i(n)$ is complex Gaussian, then [10]

$$R_{x_{\phi\phi}}(k,l) = E[x_i(n)x_i^*(n-l)]E[x_i^*(n-k)x_i(n-l-k)] \\ + E[x_i(n)x_i^*(n-k)]E[x_i^*(n-l)x_i(n-l-k)]. \tag{B.2}$$

where use has been made of the fact that $E[x_i(n)x_i(k)]=0$ for $n \neq k$. However, we do not wish to constrain this discussion to this restrictive case. Rather, we wish to consider the more general case of a Gaussian process $x_i(n)$ with unconstrained quadrature components. We therefore consider

$$x_i(n) = x_{iI}(n) + j\, x_{iQ}(n) \tag{B.3}$$

where the processes $x_{iI}(n)$ and $x_{iQ}(n)$ are jointly Gaussian. Using eq(B.3) in (B.1), we obtain

$$R_{x_{\phi\phi}}(k,l) = E\{[x_{iI}(n) + j\, x_{iQ}(n)][x_{iI}(n-l) - j\, x_{iQ}(n-l)] \\ \bullet\ [x_{iI}(n-k) - j\, x_{iQ}(n-k)][x_{iI}(n-l-k) + j\, x_{iQ}(n-l-k)]\} \tag{B.4a}$$

$$= E\{[x_{iI}(n)x_{iI}(n-l)+x_{iQ}(n)x_{iQ}(n-l)+jx_{iQ}(n)x_{iI}(n-l)-jx_{iI}(n)x_{iQ}(n-l)] \\ \bullet\ [x_{iI}(n-k)x_{iI}(n-l-k)+x_{iQ}(n-k)x_{iQ}(n-l-k) \\ +jx_{iI}(n-k)x_{iQ}(n-l-k) - jx_{iQ}(n-k)x_{iI}(n-l-k)]\} \tag{B.4b}$$

$$
\begin{aligned}
&= E[x_{iI}(n)x_{iI}(n-l)x_{iI}(n-k)x_{iI}(n-l-k)] + E[x_{iQ}(n)x_{iQ}(n-l)x_{iI}(n-k)x_{iI}(n-l-k)] \\
&+ E[x_{iI}(n)x_{iI}(n-l)x_{iQ}(n-k)x_{iQ}(n-l-k)] + E[x_{iQ}(n)x_{iQ}(n-l)x_{iQ}(n-k)x_{iQ}(n-l-k)] \\
&- E[x_{iQ}(n)x_{iI}(n-l)x_{iI}(n-k)x_{iQ}(n-l-k)] + E[x_{iI}(n)x_{iQ}(n-l)x_{iI}(n-k)x_{iQ}(n-l-k)] \\
&+ E[x_{iQ}(n)x_{iI}(n-l)x_{iQ}(n-k)x_{iI}(n-l-k)] - E[x_{iI}(n)x_{iQ}(n-l)x_{iQ}(n-k)x_{iI}(n-l-k)] \\
&+jE[x_{iI}(n)x_{iI}(n-l)x_{iI}(n-k)x_{iQ}(n-l-k)] - jE[x_{iI}(n)x_{iI}(n-l)x_{iQ}(n-k)x_{iI}(n-l-k)] \\
&+jE[x_{iQ}(n)x_{iQ}(n-l)x_{iI}(n-k)x_{iQ}(n-l-k)] - jE[x_{iQ}(n)x_{iQ}(n-l)x_{iQ}(n-k)x_{iI}(n-l-k)] \\
&+jE[x_{iQ}(n)x_{iI}(n-l)x_{iI}(n-k)x_{iI}(n-l-k)] + jE[x_{iQ}(n)x_{iI}(n-l)x_{iQ}(n-k)x_{iQ}(n-l-k)] \\
&-jE[x_{iI}(n)x_{iQ}(n-l)x_{iI}(n-k)x_{iI}(n-l-k)] - j\,E[x_{iI}(n)x_{iQ}(n-l)x_{iQ}(n-k)x_{iQ}(n-l-k)]
\end{aligned}
\tag{B.4c}
$$

For Gaussian, zero-mean quadrature components, eq(F.4c) can be expressed as

$$
\begin{aligned}
R_{x_{\phi\phi}}(k,l) = {} & [R_{ii}^{II}(l)]^2 + [R_{ii}^{II}(k)]^2 + R_{ii}^{II}(l+k)R_{ii}^{II}(k-l) \\
& + R_{ii}^{QQ}(l)R_{ii}^{II}(l) + [R_{ii}^{QI}(k)]^2 + R_{ii}^{QI}(l+k)R_{ii}^{QI}(k-l) \\
& + R_{ii}^{II}(l)R_{ii}^{QQ}(l) + [R_{ii}^{IQ}(k)]^2 + R_{ii}^{IQ}(l+k)R_{ii}^{IQ}(k-l) \\
& + [R_{ii}^{QQ}(l)]^2 + [R_{ii}^{QQ}(k)]^2 + R_{ii}^{QQ}(l+k)R_{ii}^{QQ}(k-l) \\
& - R_{ii}^{QI}(l)R_{ii}^{IQ}(l) - R_{ii}^{QI}(k)R_{ii}^{IQ}(k) - R_{ii}^{QQ}(l+k)R_{ii}^{II}(k-l) \\
& + [R_{ii}^{IQ}(l)]^2 + R_{ii}^{II}(k)R_{ii}^{QQ}(k) + R_{ii}^{IQ}(l+k)R_{ii}^{QI}(k-l) \\
& + [R_{ii}^{QI}(l)]^2 + R_{ii}^{QQ}(k)R_{ii}^{II}(k) + R_{ii}^{QI}(l+k)R_{ii}^{IQ}(k-l) \\
& - R_{ii}^{IQ}(l)R_{ii}^{QI}(l) - R_{ii}^{IQ}(k)R_{ii}^{QI}(k) - R_{ii}^{II}(l+k)R_{ii}^{QQ}(k-l) \\
& +j\left\{R_{ii}^{II}(l)R_{ii}^{IQ}(l) + R_{ii}^{II}(k)R_{ii}^{IQ}(k) + R_{ii}^{IQ}(l+k)R_{ii}^{II}(k-l)\right\} \\
& -j\left\{R_{ii}^{II}(l)R_{ii}^{QI}(l) + R_{ii}^{IQ}(k)R_{ii}^{II}(k) + R_{ii}^{II}(l+k)R_{ii}^{IQ}(k-l)\right\} \\
& +j\left\{R_{ii}^{QQ}(l)R_{ii}^{IQ}(l) + R_{ii}^{QI}(k)R_{ii}^{QQ}(k) + R_{ii}^{QQ}(l+k)R_{ii}^{QI}(k-l)\right\} \\
& -j\left\{R_{ii}^{QQ}(l)R_{ii}^{QI}(l) + R_{ii}^{QQ}(k)R_{ii}^{QI}(k) + R_{ii}^{QI}(l+k)R_{ii}^{QQ}(k-l)\right\} \\
& +j\left\{R_{ii}^{QI}(l)R_{ii}^{II}(l) + R_{ii}^{QI}(k)R_{ii}^{II}(k) + R_{ii}^{QI}(l+k)R_{ii}^{II}(k-l)\right\} \\
& +j\left\{R_{ii}^{QI}(l)R_{ii}^{QQ}(l) + R_{ii}^{QQ}(k)R_{ii}^{IQ}(k) + R_{ii}^{QQ}(l+k)R_{ii}^{IQ}(k-l)\right\} \\
& -j\left\{R_{ii}^{IQ}(l)R_{ii}^{II}(l) + R_{ii}^{II}(k)R_{ii}^{QI}(k) + R_{ii}^{II}(l+k)R_{ii}^{QI}(k-l)\right\} \\
& -j\left\{R_{ii}^{IQ}(l)R_{ii}^{QQ}(l) + R_{ii}^{IQ}(k)R_{ii}^{QQ}(k) + R_{ii}^{IQ}(l+k)R_{ii}^{QQ}(k-l)\right\}
\end{aligned}
\tag{B.5}
$$

where we note that the first two terms in each parenthesis for the imaginary terms cancel. Since

$$R(l) = [R_{ii}^{II}(l) + R_{ii}^{QQ}(l)] + j\,[R_{ii}^{QI}(l) - R_{ii}^{IQ}(l)] \tag{B.6}$$

then

$$|R(l)|^2 = [R_{ii}^{II}(l)]^2 + 2R_{ii}^{II}(l)R_{ii}^{QQ}(l) + [R_{ii}^{QQ}(l)]^2 + [R_{ii}^{QI}(l)]^2 - 2R_{ii}^{QI}(l)R_{ii}^{IQ}(l) + [R_{ii}^{IQ}(l)]^2 \tag{B.7}$$

and similarly for $|R(k)|^2$ so that

$$R_{x_{\phi\phi}}(k,l) = |R_{ii}(l)|^2 + |R_{ii}(k)|^2 + F_{ii}(l,k) \tag{B.8}$$

where

$$\begin{aligned} F_{ii}(l,k) = &\left\{R_{ii}^{II}(l+k) - R_{ii}^{QQ}(l+k)\right\}\left\{R_{ii}^{II}(l-k) - R_{ii}^{QQ}(l-k)\right\} \\ &+ \left\{R_{ii}^{QI}(l+k) + R_{ii}^{IQ}(l+k)\right\}\left\{R_{ii}^{QI}(l-k) + R_{ii}^{IQ}(l-k)\right\} \\ &- j\left\{R_{ii}^{II}(l+k) - R_{ii}^{QQ}(l+k)\right\}\left\{R_{ii}^{QI}(l-k) + R_{ii}^{IQ}(l-k)\right\} \\ &+ j\left\{R_{ii}^{II}(l-k) - R_{ii}^{QQ}(l-k)\right\}\left\{R_{ii}^{QI}(l+k) + R_{ii}^{IQ}(l+k)\right\}. \end{aligned} \tag{B.9}$$

APPENDIX C DERIVATION OF THE EXPRESSION FOR $E[s^4]$

In this Appendix, an analytic expression for $E[s^4]$ is derived in terms of the shape and scale parameters of $f_s(s)$ where $f_s(s)$ is a Gamma distribution and the random variable s is real and non-negative. In this case, the observation data vector process $\underline{y}_{1,N} = s\underline{x}_{1,N}$ described in eq(2.1) has a K-distribution. The author is indebted to Muralidhar Rangaswamy for suggestinging the following derivation. For the Gamma distributed $f_s(s)$ with non-negative s,

$$E[s^4] = \frac{2b}{\Gamma(\alpha)} \int_0^\infty (bs)^{2\alpha-1} s^4 \exp(-b^2 s^2) ds \tag{C.1a}$$

$$= \frac{2b}{\Gamma(\alpha)} \int_0^\infty (b)^{2\alpha-1} s^{2\alpha+3} \exp(-b^2 s^2) ds. \tag{C.1b}$$

Consider s=s'/b, so that

$$f_{s'}(s') = \frac{2}{\Gamma(\alpha)} (s')^{2\alpha-1} \exp[(s')^2]. \tag{C.2}$$

Now let $s'=\sqrt{w}$, so that

$$f_w(w) = \frac{w^{\alpha-1} e^{-w}}{\Gamma(\alpha)}. \tag{C.3}$$

Using s=s'/b and $s'=\sqrt{w}$,

$$E[s^4] = E[(s')^4]/b^4 = E[w^2]/b^4. \tag{C.4}$$

However,

$$E[w^2] = \int_0^\infty \frac{w^{\alpha+1} e^{-w}}{\Gamma(\alpha)} dw \tag{C.5a}$$

$$= \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} = \alpha(\alpha+1). \tag{C.5b}$$

Using (C.5b) in (C.4)

$$E[s^4] = E[w^2]/b^4 = \frac{\alpha(\alpha+1)}{b^4}. \tag{C.6}$$

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